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Iman Transform for Solving Linear Ordinary Differential Equations with Variable Coefficients

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ABSTRACT: This manuscript presents a novel integral transform, designated as the Iman transform, which is employed to resolve ordinary differential equations characterized by variable coefficients, owing to their critical significance across diverse domains, particularly within applied mathematics. The Iman transform is categorized as a contemporary integral transform, which has previously exhibited its efficacy in addressing equations with constant coefficients. This investigation elucidates the operational attributes of the transform and accentuates its capacity to convert differential equations with variable coefficients into more manageable forms. In the course of this research, explicit formulations of the Iman transform are established for specific functions $t^n f(t), t^n f'(t), t^n f''(t), n \in Z^+$, and these formulations are utilized to tackle initial value problems featuring variable coefficients. The results indicate that the Iman transform furnishes a precise and dependable methodology for deriving analytical solutions, obviating the necessity for conventional approaches and highlighting its importance as a formidable instrument in the analysis and resolution of this category of differential equations.

Keywords: Iman Transform, Linear Ordinary Differential Equations, Variable Coefficients, Integral Transforms, Applied Mathematics.

الملخص:

في هذا البحث، تم استخدام تحويل تكاملي جديد، يسمى تحويل إيمان (**Iman Transform**)، لحل المعادلات التفاضلية العادية ذات المعاملات المتغيرة نظراً لتطبيقها الكبير في مختلف المجالات، لاسيما في الرياضيات التطبيقية. يعتبر تحويل إيمان من التحويلات التكاملية الحديثة التي أُثبتت سابقاً فعاليته في حل المعادلات التفاضلية ذات المعاملات الثابتة. يوضح هذا البحث الخصائص التشغيلية لتحويل إيمان ويتحقق من قدرته على تحويل المعادلات التفاضلية ذات المعاملات المتغيرة إلى أشكال قابلة للتحليل. كجزء من هذا البحث، تم اشتقاق صيغ صريحة لتحويل إيمان للدوال $t^n f(t), t^n f'(t), t^n f''(t), n \in Z^+$ ، وتم تطبيق هذه الصيغ على مسائل قيم ابتدائية ذات معاملات متغيرة. تشير النتائج إلى أن تحويل إيمان يقدم منهجية دقيقة وموثوقة لاشتقاق الحلول التحليلية للمعادلات ذات المعاملات المتغيرة، دون الحاجة إلى الطرق التقليدية، مما يؤكد أهميته كأداة فعالة في حل هذه الفئة من المعادلات التفاضلية.

الكلمات المفتاحية: تحويل إيمان، المعادلات التفاضلية العادية الخطية، المعاملات المتغيرة، التحويلات التكاملية، الرياضيات التطبيقية.

I. INTRODUCTION

Integral transforms constitute essential methodologies in the realm of applied mathematics, particularly in the context of differential equations. These mathematical constructs facilitate the resolution of ordinary, partial, and integral differential equations with greater efficacy compared to conventional techniques. Since the inception of the Laplace transform in 1780 (Spiegel, 1965), succeeded by the Fourier transform, these transforms have assumed a pivotal role in addressing differential equations that emerge across diverse scientific domains, including physics, engineering, and associated fields. In light of the criticality of integral transforms, recent years have observed a burgeoning interest in the innovation of novel transforms, the refinement of existing ones, and the augmentation of their practical applications. Numerous scholarly articles have been published in this domain, notably presenting transforms such as the Kamal transform (Kamal et al., 2016) and the Mohand transform (Mahgoub, 2017), which are derived from the Fourier transform, among others. These transforms have been extensively utilized to solve ordinary and partial differential equations characterized by constant coefficients. Moreover, the Anuj and Mahgoub transforms (Kumar, 2021; Mahgoub, 2016) have made significant contributions to the resolution of ordinary differential equations with constant coefficients. Furthermore, the Mohand transform (Attaweel et al., 2020), the Anuj transform (Rashdi, 2022), and the SEE transform (Mehdi, 2023) have been introduced to tackle ordinary differential equations with variable coefficients. Collectively, these transforms have demonstrated their efficacy in deriving exact solutions. Nevertheless, there persists a necessity for integral transforms adept at addressing specific categories of ordinary differential equations with variable coefficients.

The Iman transform was initially proposed by Khalid Iman and subsequently formalized and examined by Almardy, I. A. (2023). The Iman transform is delineated for functions of exponential order, wherein we consider functions within the set: $A = \{\varphi(t): \exists M, k_1, k_2 > 0, |\varphi(t)| < Me^{-v^2 t}\}$, characterized by the stipulation that the constant M must be a finite quantity, k_1, k_2 , which may be either finite or infinite.

The Iman transform of the function $\varphi(t)$ is articulated as follows:

$$I[\varphi(t)] = \frac{1}{v^2}\int_0^\infty \varphi(t)\, e^{-v^2 t} dt = \Phi(v), t \geq 0, k_1 \leq v \leq k_2.$$

Where (I) the operator signifies the Iman transform, the variable (v) in this transform represents the argument of the function φ. Almardy (2023) employed the Iman transform for the resolution of linear ordinary differential equations with constant coefficients. Almardy et al. (2023) conducted a comparative analysis between the Iman and Laplace transforms for the resolution of ordinary differential equations of both first and second order. Almardy et al. (2024) applied this transform for the resolution of partial differential equations.

The objective of this research is to make a significant contribution toward addressing this methodological deficiency by proposing the Iman Transform as a novel analytical instrument. This transform is centered on leveraging novel mathematical attributes to proficiently manage the coefficient variables, thus offering a route to resolving linear ordinary differential equations with variable coefficients with enhanced efficiency and precision. This study will rigorously establish the mathematical foundation of the Iman Transform, implement it in practical scenarios, and assess its effectiveness in comparison to alternative methodologies, thereby positioning it as a substantial enhancement to the toolkit of differential equations.

II. Iman Transform and Invers Iman Transform of Some elementary Functions (Almardy, 2023., Khalat,2025).

The subsequent Iman transform and its corresponding inverse Iman transform for various functions are delineated in tables (1) and (2) (Almardy, 2023). Moreover, these formulas are derived from the identical source.

Table 1. Iman transforms of some elementary functions.

S.N	$\varphi(t)$	$I[\varphi(t)] = \Phi(v)$
1	1	$\frac{1}{v^4}$
2	k	$\frac{k}{v^4}$
3	t	$\frac{1}{v^6}$
4	$t^n, n \in N$	$\frac{n!}{v^{2n+4}}$
5	$t^n, n > -1$	$\frac{\Gamma(n+1)}{v^{2n+4}}$
6	e^{at}	$\frac{1}{v^4 - av^2}$
7	e^{-at}	$\frac{1}{v^4 + av^2}$
8	$sin(\alpha t)$	$\frac{\alpha}{v^2(v^4 + \alpha^2)}$
9	$cos(\alpha t)$	$\frac{1}{v^4 + \alpha^2}$
10	$sinh(\alpha t)$	$\frac{\alpha}{v^2(v^4 - \alpha^2)}$
11	$cosh(\alpha t)$	$\frac{1}{v^4 - \alpha^2}$

If $I[\varphi(t)] = \Phi(v)$ then $\varphi(t)$ is called the inverse Iman transform of $\Phi(v)$ and mathematically it is defined as $\varphi(t) = I^{-1}[\Phi(v)]$ where I^{-1} is the inverse Iman transform operator.

Table 2. Inverse Iman transforms of some elementary functions.

S.N	$\Phi(v)$	$\varphi(t) = I^{-1}[\Phi(v)]$
1	$\frac{1}{v^4}$	1
2	$\frac{k}{v^4}$	k
3	$\frac{1}{v^6}$	t
4	$\frac{n!}{v^{2n+4}}, n \in N$	t^n
5	$\frac{\Gamma(n+1)}{v^{2n+4}}, n > -1$	t^n
6	$\frac{1}{v^4 - av^2}$	e^{at}

7	$\frac{1}{v^4+av^2}$	e^{-at}
8	$\frac{\alpha}{v^2(v^4+\alpha^2)}$	$sin(\alpha t)$
9	$\frac{1}{v^4+\alpha^2}$	$cos(\alpha t)$
10	$\frac{\alpha}{v^2(v^4-\alpha^2)}$	$sinh(\alpha t)$
11	$\frac{1}{v^4-\alpha^2}$	$cosh(\alpha t)$

III. TRANSLATION PROPTY OF IMAN TRANSFORM (Khalat. 2025).

If; $I[\varphi(t)] = \Phi(v)$ then; $I[e^{kt}\varphi(t)] = \frac{v^2-k}{v^2}\ \Phi\left(\sqrt{v^2-k}\right)$.

Proof: By the definition of Iman transform, we have

$$I[\varphi(t)] = \frac{1}{v^2}\int_0^\infty \varphi(t)\, e^{-v^2 t}dt = \Phi(v), \text{ then;}$$

$$I[e^{kt}\varphi(t)] = \frac{1}{v^2}\int_0^\infty e^{kt}\varphi(t)\, e^{-v^2 t}dt$$

$$= \frac{1}{v^2}\int_0^\infty \varphi(t)\, e^{-(v^2-k)\,t}dt$$

$$= \frac{\left(\sqrt{v^2-k}\right)^2}{v^2}\left[\frac{1}{\left(\sqrt{v^2-k}\right)^2}\int_0^\infty \varphi(t)\, e^{-(v^2-k)\,t}dt\right]$$

$$= \frac{v^2-k}{v^2}\,\Phi\left(\sqrt{v^2-k}\right)$$

IV. IMAN TRANSFORM OF THE DERIVATIVES OF THE FUNCTION $\varphi(t)$ (Almardy, 2023)

If $I[\varphi(t)] = \Phi(v)$ then; (1)

A) $I[\varphi'(t)] = v^2\Phi(v) - \frac{1}{v^2}\ \varphi(0)$ (2)

B) $I[\varphi''(t)] = v^4\Phi(v) - \varphi(0) - \frac{1}{v^2}\ \varphi'(0)$ (3)

C) $I\left[\varphi^{(n)}(t)\right] = v^{2n}\Phi(v) - \sum_{k=0}^{n-1}\frac{1}{v^{4-2n-2k}}\ \varphi^{(k)}(0)$ (4)

V. IMAN TRANSFORM OF $t\varphi(t),\ t^2\varphi(t),\ t^3\varphi(t)$:

Let a function $\varphi(t)$ and $I[\varphi(t)] = \Phi(v)$ then:

A) $I[\,t\,\varphi(t)\,] = \frac{-1}{2v}\frac{d}{dv}\Phi(v) - \frac{1}{v^2}\ \Phi(v)$

B) $I[t^2\varphi(t)] = \frac{1}{4v^2}\frac{d^2}{dv^2}\Phi(v) + \frac{3}{4v^3}\frac{d}{dv}\Phi(v)$

C) $I[t^3\varphi(t)] = \frac{-1}{8v^3}\frac{d^3}{dv^3}\Phi(v) - \frac{3}{8v^4}\frac{d^2}{dv^2}\Phi(v) + \frac{3}{8v^5}\frac{d}{dv}\Phi(v)$

Proof:

A). From the definition of Iman transform we have:

$$I[\varphi(t)] = \Phi(v) = \frac{1}{v^2}\int_0^\infty \varphi(t)\, e^{-v^2 t}dt$$

Differentiating the above equations, we obtain:

$$\frac{d}{dv}\, I[\,\varphi(t)] = \frac{d}{dv}\left[\frac{1}{v^2}\int_0^\infty \varphi(t)\, e^{-v^2 t}dt\right]$$

$$= \frac{-2}{v}\int_0^\infty t\,\varphi(t)\, e^{-v^2 t}dt - \frac{2}{v^3}\int_0^\infty \varphi(t)\, e^{-v^2 t}dt$$

$$= -2v\, I[t\,\varphi(t)] - \frac{2}{v}\,\Phi(v)$$

$$\Rightarrow\; I[t\,\varphi(t)] = \frac{-1}{2v}\frac{d}{dv}\Phi(v) - \frac{1}{v^2}\,\Phi(v) \qquad (5)$$

B). By replacing tφ(t) with φ(t) in equation (5) we obtain**:**

$$I[\,t^2\,\varphi(t\,)] \;=\; \frac{-1}{2v}\frac{d}{dv}\, I[t\,\varphi(t)] - \frac{1}{v^2}\;I[t\,\varphi(t)]$$

$$= \frac{-1}{2v}\frac{d}{dv}\left[\frac{-1}{2v}\frac{d}{dv}\Phi(v) - \frac{1}{v^2}\,\Phi(v)\right] - \frac{1}{v^2}\left[\frac{-1}{2v}\frac{d}{dv}\Phi(v) - \frac{1}{v^2}\Phi(v)\right]$$

$$\Rightarrow I[t^2\varphi(t)] = \frac{1}{4v^2}\frac{d^2}{dv^2}\Phi(v) + \frac{3}{4v^3}\frac{d}{dv}\Phi(v) \qquad (6)$$

C). To find $I[t^3\varphi(t)]$ we putting $t^2\varphi(t)$ in equation (5) instead of φ(t), and by use equation (6) we obtain*:*

$$I[t^3\varphi(t)] = \frac{-1}{2v}\frac{d}{dv}\, I[t^{\,2}\varphi(t)] - \frac{1}{v^2}\; I[t^2\,\varphi(t)]$$

$$= \frac{-1}{2v}\frac{d}{dv}\left[\frac{1}{4v^2}\frac{d^2}{dv^2}\Phi(v) + \frac{3}{4v^3}\frac{d}{dv}\Phi(v)\right] - \frac{1}{v^2}\left[\frac{1}{4v^2}\frac{d^2}{dv^2}\Phi(v) + \frac{3}{4v^3}\frac{d}{dv}\Phi(v)\right]$$

$$\Rightarrow I[t^3\varphi(t)] = \frac{-1}{8v^3}\frac{d^3}{dv^3}\Phi(v) - \frac{3}{8v^4}\frac{d^2}{dv^2}\Phi(v) + \frac{3}{8v^5}\frac{d}{dv}\Phi(v)$$

VI. IMAN TRANSFORM OF $t\varphi'(t),\ t^2\varphi'(t),\ t^3\varphi'(t)$:

If $I[\varphi(t)] = \Phi(v)$ then:

A). $I[t\varphi'(t)] = \frac{-v}{2}\frac{d}{dv}\Phi(v) - 2\Phi(v)$

B). $I[t^2\varphi'(t)] = \frac{1}{4}\frac{d^2}{dv^2}\Phi(v) + \frac{7}{4v}\frac{d}{dv}\Phi(v) + \frac{2}{v^2}\ \Phi(v)$

C). $I[t^3\varphi'(t)] = \frac{-1}{8v}\frac{d^3}{dv^3}\Phi(v) - \frac{9}{8v^2}\frac{d^2}{dv^2}\Phi(v) - \frac{15}{8v^3}\ \Phi(v)$

Proof:

A). In equation (5) put $\varphi'(t)$ replace $\varphi(t)$ we get:

$$I[t\ \varphi'(t)] = \frac{-1}{2v}\frac{d}{dv}I[\varphi'(t)] - \frac{1}{v^2}\ I[\varphi'(t)]. \text{ Since, } I[\varphi'(t)] = v^2\Phi(v) - \frac{1}{v^2}\ \varphi(0)$$

$$\Rightarrow\ I[t\ \varphi'(t)] = \frac{-1}{2v}\frac{d}{dv}\left[v^2\Phi(v) - \frac{1}{v^2}\ \varphi(0)\right] - \frac{1}{v^2}\left[v^2\Phi(v) - \frac{1}{v^2}\ \varphi(0)\right]$$

$$= \frac{-v}{2}\frac{d}{dv}\Phi(v) - 2\Phi(v) + \frac{1}{2v^3}\frac{d}{dv}f(0).$$

Since, $\frac{d}{dv}f(0) = 0$. So, obtain:

$$I[t\ \varphi'(t)] = \frac{-v}{2}\frac{d}{dv}\Phi(v) - 2\Phi(v).$$

B). By replacing $\varphi(t)$ in equation (5) with $t\ \varphi'(t)$ yields:

$$I[t^2\varphi'(t)] = \frac{-1}{2v}\frac{d}{dv}I[t\ \varphi'(t)] - \frac{1}{v^2}I[t\ \varphi'(t)]$$

$$= \frac{-1}{2v}\frac{d}{dv}\left[\frac{-v}{2}\frac{d}{dv}\Phi(v) - 2\Phi(v)\right] - \frac{1}{v^2}\left[\frac{-v}{2}\frac{d}{dv}\Phi(v) - 2\Phi(v)\right]$$

$$\Rightarrow I[t^2\varphi'(t)] = \frac{1}{4}\frac{d^2}{dv^2}\Phi(v) + \frac{7}{4v}\frac{d}{dv}\Phi(v) + \frac{2}{v^2}\ \Phi(v).$$

C). The transform $I[t^3\varphi'(t)]$ is obtained by following the same steps as in the previous sections.

VII. IMAN TRANSFORM OF $t\varphi''(t),\ t^2\varphi''(t)$:

Similarly, the following transforms can be obtained by applying the same procedure as in the previous sections:

A). $I[t\varphi''(t)] = \frac{-v^3}{2}\frac{d}{dv}\Phi(v) - 3v^2\Phi(v) + \frac{1}{v^2}\ \varphi(0).$

B). $I[t^2\varphi''(t)] = \frac{v^2}{4}\frac{d^2}{dv^2}\Phi(v) + \frac{11v}{4}\frac{d}{dv}\Phi(v) + 6\Phi(v).$

Proof:

A). Replacing $\varphi(t)$ by $\varphi''(t)$ in equation (5), we have

$$I[t\ \varphi''(t)] = \frac{-1}{2v}\frac{d}{dv}I(\varphi''(t)) - \frac{1}{v^2}\ I(\varphi''(t))$$

$$= \frac{-1}{2v}\frac{d}{dv}\left[v^4\Phi(v) - \varphi(0) - \frac{1}{v^2}\ \varphi'(0)\right] - \frac{1}{v^2}\left[v^4\Phi(v) - \varphi(0) - \frac{1}{v^2}\ \varphi'(0)\right]$$

On simplification, we have

$$I[t\ \varphi''(t)] = \frac{-v^3}{2}\frac{d}{dv}\Phi(v) - 3v^2\Phi(v) + \frac{1}{v^2}\ \varphi(0).$$

B). Replacing $\varphi(t)$ by $t\varphi''(t)$ in equation (5), we have:

$$I[t^{\ 2}\varphi''(t)] = \frac{-1}{2v}\frac{d}{dv}I(t\varphi''(t)) - \frac{1}{v^2}\ I(t\varphi''(t))$$

$$= \frac{-1}{2v}\frac{d}{dv}\left[\frac{-v^3}{2}\frac{d}{dv}\Phi(v) - 3v^2\Phi(v) + \frac{1}{v^2}\ \varphi(0)\right] - \frac{1}{v^2}\ \left[\frac{-v^3}{2}\frac{d}{dv}\Phi(v) - 3v^2\Phi(v) + \frac{1}{v^2}\ \varphi(0)\right]$$

Simplifying the above expression, we have;

$$I[t^2\varphi''(t)] = \frac{v^2}{4}\frac{d^2}{dv^2}\Phi(v) + \frac{11v}{4}\frac{d}{dv}\Phi(v) + 6\Phi(v).$$

Notice: The Iman integral transform of $t^n\varphi^{(m)}(t), n, m\ \in N$ can be determined by following the same procedure.

VIII. APPLICATIONS

This section presents applications of the Iman transform in solving ordinary differential equations with variable coefficients.

Application (1). Consider the linear ordinary differential equations with variable coefficients:

$$tu'' + u' = 4t^2, \qquad u(0) = 0$$

Applying Iman transform to both sides of the given equation yields:

$$I(tu'') + I(u') = 4I(t^2)$$

$$\frac{-v^3}{2}\frac{d}{dv}\Phi(v) - 3v^2\Phi(v) + \frac{1}{v^2}\ \varphi(0) + v^2\Phi(v) - \frac{1}{v^2}\ \varphi(0) = 4\left(\frac{2!}{v^8}\right)$$

Using the given initial condition and simplifying above equation, we have:

$$\frac{d}{dv}\Phi(v) + \frac{4}{v}\Phi(v) = \frac{-16}{v^{11}}$$

This is a first order linear ordinary differential equation in the unknown $\Phi(v)$. Therefore:

$$\Phi(v) = e^{-\int\frac{4}{v}dv}\left[\int \frac{-16}{v^{11}} e^{\int\frac{4}{v}dv}\ dv + c\right]$$

$$\Rightarrow \Phi(v) = \frac{8}{3}\frac{1}{v^{10}} + \frac{c}{v^4}$$

By taking inverse Iman transform for both sides, we obtain:

$$u(t) = \frac{8}{3}I^{-1}\left(\frac{1}{v^{10}}\right) + cI^{-1}\left(\frac{1}{v^4}\right) = \frac{4}{9}t^3 + c$$

Application (2). Consider the linear ordinary differential equations with variable coefficients

$$tu'' - tu' - u = 0, \qquad u(0) = 0,\ u'(0) = 2$$

Applying Iman transform to both sides of the given equation yields:

$$I(tu'') - I(tu') - I(u) = O$$

$$\frac{-v^3}{2}\frac{d}{dv}I(u) - 3v^2 I(u) + \frac{1}{v^2}u(0) + \frac{v}{2}\frac{d}{dv}I(u) + 2I(u) - I(u) = 0$$

By simplifying the above expression, we obtain:

$$\frac{v - v^3}{2}\frac{d}{dv}I(u) = (3v^2 - 1)I(u)$$

$$\Rightarrow \int \frac{dI(u)}{I(u)} = \int \frac{-2(3v^2 - 1)}{v^3 - v}dv$$

$$\Rightarrow ln\ I(u)\ = ln\ \frac{c}{(v^3 - v)^2}$$

$$\Rightarrow I(u) = \frac{c}{(v^3 - v)^2}$$

By taking inverse Iman transform for both sides, we obtain:

$$u(t) = c\, I^{-1}\left(\frac{1}{(v^3 - v)^2}\right) = c\, I^{-1}\left(\frac{1}{v^2(v^2 - 1)^2}\right) = ce^t t.$$

From the second initial condition: $u'(0) = 2 \Rightarrow c = 2$

Then the solution can be expressed as:

$$u(t) = 2e^t t$$

Application (3). Consider the linear ordinary differential equations with variable coefficients

$$tu'' + (1 - 2t)u' - 2u = 0, \qquad u(0) = 1, u'(0) = 2$$

By applying Iman transform to given equation, we have:

$$I(tu'') + I[(1 - 2t)u'] - 2I(u) = 0$$

$$I(tu'') + I(u') - 2I(tu') - 2I(u) = 0$$

$$\frac{-v^3}{2}\frac{d}{dv}\Phi(v) - 3v^2\Phi(v) + \frac{1}{v^2}u(0) + v^2\Phi(v) - \frac{1}{v^2}u(0) - 2\left[\frac{-v}{2}\frac{d}{dv}\Phi(v) - 2\Phi(v)\right] - 2\Phi(v) = 0$$

By using given initial conditions and simplifying the above expression, we obtain:

$$\frac{d\Phi(v)}{\Phi(v)} = \frac{4v^2 - 4}{2v - v^3}dv$$

$$\Rightarrow ln\,\Phi(v) = \int \frac{4v^2 - 4}{2v - v^3}dv$$

$$\Rightarrow ln\,\Phi(v) = \int \frac{-2}{v}dv + \int \frac{2v}{2 - v^2}dv$$

$$\Rightarrow \Phi(v) = \frac{c}{2v^2 - v^4} = \frac{C}{v^4 - 2v^2}, \quad C = -c$$

Now, by applying inverse Iman transform, we obtain:

$$u(t) = I^{-1}\left[\frac{C}{v^4 - 2v^2}\right] = CI^{-1}\left[\frac{1}{v^4 - 2v^2}\right] = Ce^{2t}$$

From the second initial condition: $u'(0) = 2 \Rightarrow C = 1$

Therefore, the solution is: $u(t) = e^{2t}$.

Application (4). Consider the linear ordinary differential equations with variable coefficients

$$t^2u'' - tu' + u = 5, \qquad u(0) = 5, u'(0) = 3$$

$$I(t^2u'') - I(tu') + I(u) = I(5)$$

$$\frac{v^2}{4}\frac{d^2}{dv^2}\Phi(v) + \frac{11v}{4}\frac{d}{dv}\Phi(v) + 6\Phi(v) + \frac{v}{2}\frac{d}{dv}\Phi(v) + 2\Phi(v) + \Phi(v) = \frac{5}{v^4}$$

$$\Rightarrow \frac{v^2}{4}\frac{d^2}{dv^2}\Phi(v) + \frac{13v}{4}\frac{d}{dv}\Phi(v) + 9\Phi(v) = \frac{5}{v^4}$$

Let $N = \Phi(v)$

$$\Rightarrow \frac{v^2}{4}\frac{d^2N}{dv^2} + \frac{13v}{4}\frac{dN}{dv} + 9N = \frac{5}{v^4}$$

This is the Cauchy-Euler differential equation. Therefore, we will consider:

$v = e^x \Rightarrow x = ln\, v \ \Rightarrow \frac{dN}{dv} = e^{-x}\frac{dN}{dx}, \frac{d^2N}{dv^2} = e^{-2x}\left(\frac{d^2N}{dx^2} - \frac{dN}{dx}\right)$

By substituting into above the Cauchy-Euler differential equation, we obtain:

$$\frac{e^{2x}}{4}\left[e^{-2x}\left(\frac{d^2N}{dx^2} - \frac{dN}{dx}\right)\right] + \frac{13e^x}{4}\left[e^{-x}\frac{dN}{dx}\right] + 9N = \frac{5}{e^{4x}}$$

By simplifying the above equation, we obtain:

$$\frac{d^2N}{dx^2} + 12\frac{dN}{dx} + 36N = 20e^{-4x}$$

This is non-homogeneous second- order differential equation with constant coefficients and its solution is of the form:

$$N_g = N_c + N_p = c_1e^{-6x} + c_2xe^{-6x} + 5e^{-4x}$$

$$\Rightarrow \Phi(v) = \frac{c_1}{v^6} + \frac{c_2 ln\, v}{v^6} + \frac{5}{v^4}$$

Requiring $y(0)$ to be finite yields $c_2 = 0$

$$\Rightarrow \Phi(v) = \frac{c_1}{v^6} + \frac{5}{v^4}$$

By applying inverse Iman transform, we obtain:

$$u(t) = I^{-1}\left(\frac{c_1}{v^6}\right) + I^{-1}\left(\frac{5}{v^4}\right) = c_1t + 5$$

Since, $u'(0) = 3 \Rightarrow u' = c_1 \Rightarrow c_1 = 3$

$$\Rightarrow u(t) = 3t + 5$$

IX. CONCLUSIONS

In the present study, the Iman Transform has been adeptly formulated and rigorously substantiated as a novel integral transform, specifically devised to confront the complexities associated with Linear Ordinary Differential Equations characterized by Variable Coefficients. Transform formulas for the functions $t^n f(t), t^n f'(t), t^n f''(t), n \in Z^+$, were derived and subsequently employed in initial value problems that involve variable coefficients. The findings illustrated the efficacy of the transform in yielding precise solutions, which were corroborated through their substitution back into the original equations and juxtaposition with solutions derived utilizing the Laplace transform and other integral transforms. This investigation suggests that the Iman transform possesses the potential for further extension in the future, allowing for a comparative analysis between the Iman transform and alternative integral transforms in the resolution of ordinary differential equations with variable coefficients.

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