The first order cyclic cohomology group of some commutative semigroup algebras

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ملخص باللغة العربية

في هذه الورقة سنحسب زمرة الكوهومولوجي الدائرية من الرتبة الأولى$\mathcal{HC}^1(\ell^1(S),\ell^\infty(S))$ حيث S هي شبه زمرة تبديلية وقابلة للاختصار صفريا ومرفقة بعنصر وحدة.

Abstract

In this paper, we shall calculate the first order cyclic cohomology group $\mathcal{HC}^1(\ell^1(S),\ell^\infty(S))$ where S is a certain commutative, 0-cancellative, $nil^\sharp$-semigroup.

Keywords: semigroup, nil-semigroup, 0-cancellative semigroup, semigroup algebra, cyclic cohomology group.

1 Introduction

We follow (Dales, 2000) and (Ghlaio, 2018) to recall some definitions and some preliminaries regarding the theory of Banach Algebras, Cyclic

Cohomology and the commutative semigroup algebras $\ell^1(S)$, where S is a commutative, 0-cancellative, $nil^\sharp$-semigroup, as also introduced in (Read, 2011).

Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. A linear map $D: \mathcal{A} \to X$ is a *derivation* if it satisfies the equation:

$$D(ab) = a \cdot Db + Da \cdot b \quad (a, b \in \mathcal{A}) .$$

In this paper we shall only consider bounded derivations. Given $x \in X$ and define the map $\delta_x: \mathcal{A} \to X$ by the equation:

$$\delta_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

These derivations are *inner* derivations.

Let X^* be the *dual space* of X. Then X^* is a Banach $\mathcal{A}$-bimodule with respect to the operations given by

$$\langle x, a \cdot \lambda \rangle = \langle x \cdot a, \lambda \rangle \quad and \quad \langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle \quad (a \in \mathcal{A}, x \in X, \lambda \in X^*) .$$

A Banach algebra $\mathcal{A}$ is *amenable* if every bounded derivation D from $\mathcal{A}$ into a dual Banach $\mathcal{A}$-bimodule X^* is inner, for each Banach $\mathcal{A}$-bimodule X. A Banach algebra $\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule over itself. Then a Banach algebra $\mathcal{A}$ is *weakly amenable* if every bounded derivation $D: \mathcal{A} \to \mathcal{A}^*$ is inner.

A linear map $T: \mathcal{A} \to \mathcal{A}^*$ is cyclic if $T(a_1)(a_0) = (-1)T(a_0)(a_1)$ for all

$a_0, a_1 \in \mathcal{A}$; in other words, $\langle a_0, T(a_1)\rangle + \langle a_1, T(a_0)\rangle = 0 \quad (a_0, a_1 \in \mathcal{A})$. In particular, $\langle a, T(a)\rangle = 0 \quad (a \in \mathcal{A})$.

The space of all bounded, cyclic derivations from $\mathcal{A}$ to $\mathcal{A}^*$ is denoted by $\mathcal{ZC}^{\,1}(\mathcal{A}, \mathcal{A}^*)$, and the set of all cyclic inner derivations from $\mathcal{A}$ to $\mathcal{A}^*$ is denoted by $\mathcal{NC}^{\,1}(\mathcal{A}, \mathcal{A}^*)$. It can be seen that every inner derivation is cyclic, and so $\mathcal{NC}^{\,1}(\mathcal{A}, \mathcal{A}^*) = \mathcal{N}^{\,1}(\mathcal{A}, \mathcal{A}^*)$. The *first-order cyclic cohomology group* is defined by

$$\mathcal{HC}^{\,1}(\mathcal{A}, \mathcal{A}^*) = \mathcal{ZC}^{\,1}(\mathcal{A}, \mathcal{A}^*)/\mathcal{NC}^{\,1}(\mathcal{A}, \mathcal{A}^*) = \mathcal{ZC}^{\,1}(\mathcal{A}, \mathcal{A}^*)/\mathcal{N}^{\,1}(\mathcal{A}, \mathcal{A}^*) .$$

Let S be a non-empty set, and let s be an element of S. The characteristic function of $\{s\}$ is denoted by δ_s. We define the *Banach space*

$$\ell^{\,1}(S) := \{f: S \to \mathbb{C}, \quad f = \textstyle\sum_{s\in S} \alpha_s \delta_s \,, \sum_{s\in S} |\alpha_s| < \infty\} ,$$

where $\|f\| = \sum_{s\in S} |\alpha_s| < \infty$.

The *dual space* of $\mathcal{A} = \ell^{\,1}(S)$ is $\mathcal{A} \;\; = \ell^{\,\infty}(S)$, where

$$\ell^{\,\infty}(S) = \left\{f: S \to \mathbb{C}, \quad \|f\| = \sup_{s\in S} |f(s)| < \infty\right\},$$

with the duality given by:

$$\langle f, \lambda\rangle = \textstyle\sum_{s\in S} f(s)\lambda(s) \quad (f \in \ell^{\,1}(S), \lambda \in \ell^{\,\infty}(S)) .$$

Let S be a semigroup. Then the *convolution product* of two elements f and g in the Banach space $\ell^1(S)$ is defined by the formula:

$$f * g = (\sum_{s\in S} \alpha_s \delta_s) * (\sum_{t\in S} \beta_t \delta_t) = \sum \{(\sum_{st=r\in S} \alpha_s \beta_t)\delta_r\}.$$

The inner sum will vanish if there are no s and t such that $st = r$.

Clearly, $(\ell^1(S),*)$ is a Banach algebra; it is called the *semigroup algebra* of S.

We shall need to use the following remark:

Remark 1.1 *Let S be a semigroup, and take g to be a function on $S \times S$. For $a, b \in S$, define*

$$T_g(\delta_a, \delta_b) = g(a, b),$$

and then extend T_g to be a bilinear function on $\ell_0^1(S) \times \ell_0^1(S)$. In the case where g is bounded by M, T_g extends to a bounded, bilinear functional on $\ell^1(S) \times \ell^1(S)$.

Explicitly,

$$\left|T_g\left(\sum_i \alpha_i \delta_{a_i}, \sum_j \beta_j \delta_{b_j}\right)\right| = \left|\sum_{i,j} \alpha_i \beta_j g(a_i, b_j)\right| \leq M \sum_i |\alpha_i| \sum_j |\beta_j|.$$

Now define $\widetilde{T_g}: \ell^1(S) \to \ell^\infty(S)$ by

$$\langle h, \widetilde{T_g}(f)\rangle = T_g(f, h) \quad (f, h \in \ell^1(S)).$$

Then $\widetilde{T_g}$ is a bounded linear map and

$$\langle \delta_b, \widetilde{T}_g(\delta_a)\rangle = g(a,b) \quad (a,b \in S).$$

Throughout the paper, S denotes a countable commutative $nil^{\sharp}$-semigroup which is the unitization of a nil semigroup S^{-} (that is, a semigroup S^{-} with zero such that for all $x \in S^{-}$, there is an $n \in \mathbb{N}$ such that $x^n = o$), and which is zero-cancellative (that is, for all $a,b,c \in S$, $ab = ac \neq o$ implies $b = c$).

Let S be the semigroup $T_n = \{e, a, a^2, \dots, a^{n-1}, a^n = o\}$ for $n \in \mathbb{N}$ with $n \geq 2$. We note that T_n is finite, commutative, 0-cancellative, $nil^{\sharp}$-semigroup. From now on we fix the notation $\mathcal{A}_n$ for the semigroup algebra $\ell^1(T_n)$.

In our paper we shall prove that $\mathcal{HC}^1(\mathcal{A}_n, \mathcal{A}_n^*) = \{0\}$.

Lemma 1.2 *Let $D: \mathcal{A}_n \to \mathcal{A}_n^*$ be a derivation. Then*

$$D(\delta_a) = \lambda_e \delta_e^* + \lambda_1 \delta_a^* + \dots + \lambda_{n-2} \delta_{a^{n-2}}^* . \qquad (1.1)$$

for some $\lambda_e, \lambda_1, \dots, \lambda_{n-2} \in \mathbb{C}$. Each such D gives a unique derivation.

Proof Note that for $r < n$ and for $a, b \in S$, we have

$$\langle \delta_b, \delta_{a^k} \cdot \delta_{a^r}^* \rangle = \langle \delta_{a^k \cdot b}, \delta_{a^r}^* \rangle = \begin{cases} 0 & if \;\; r < k \\ \langle \delta_b, \delta_{a^{r-k}}^* \rangle & if \;\; r \geq k \end{cases}$$

Indeed we see that

$$\delta_a^k \cdot \delta_{a^r}^* = \begin{cases} 0 & if \;\; r < k \\ \delta_{a^{r-k}}^* & if \;\; r \geq k \end{cases}$$

When $r = n$, we have that $\delta_{a^k} \cdot \delta_o^* = \delta_o^* + \delta_{a^{n-1}}^* + \cdots + \delta_{a^{n-k}}^*$.

Suppose that $D: \mathcal{A}_n \to \mathcal{A}_n^*$ is a derivation (automatically continuous because $\mathcal{A}_n$ is finite dimensional).

So we see that $D(\delta_e) = 0$ because δ_e is an idempotent. If $D(\delta_a)$ is as in (1.1), then we will have (for all $k > 0$)

$$D(\delta_{a^k}) = k\delta_{a^{k-1}} \cdot D(\delta_a)$$
$$= k\delta_{a^{k-1}} \cdot (\lambda_e \delta_e^* + \cdots + \lambda_{n-2}\delta_{a^{n-2}}^*)$$
$$= k \cdot \sum_{r=k-1}^{n-2} \lambda_r \delta_{a^{r-k+1}}^* .$$

These equations are consistent, because when $k \geq n$ we have that $D(\delta_{a^k}) = 0 = D(\delta_o)$.

We must have that $D(\sum_{k=0}^{n} \lambda_k \delta_{a^k}) = \sum_{k=0}^{n} \lambda_k D(\delta_{a^k})$. We then see that

$$D(\delta_{a^k} \cdot \delta_{a^l}) = D(\delta_{a^{k+l}}) = (k+l)\delta_{a^{k+l-1}} \cdot D(\delta_a) = \delta_{a^k} \cdot D(\delta_{a^l}) + D(\delta_{a^k}) \cdot \delta_{a^l}$$

for all $k, l > 0$ and in fact even when k or l is zero. So D is a derivation.

Conversely, if $D: \mathcal{A}_n \to \mathcal{A}_n^*$ is a derivation then the most general conceivable form for $D(\delta_a)$ is $D(\delta_a) = \lambda_e \delta_e^* + \sum_{r=1}^{n} \lambda_r \delta_{a^r}^*$ however we will need (since $\delta_{a^n} = \delta_o$ is idempotent)

$$0 = D(\delta_{a^n}) = n\delta_{a^{n-1}} \cdot D(\delta_a) = \lambda_{n-1}\delta_e^* + \lambda_n(\delta_o^* + \delta_a^* + \cdots + \delta_{a^{n-1}}^*)$$

so $\lambda_{n-1} = \lambda_n = 0$.

Thus the lemma is proved.

2 The main result

Now we shall prove our result in the following Theorem

Theorem 2.1 *Let* $\mathcal{A}_n = \ell^1(T_n)$, *where* $n \geq 2$. *Then* $\mathcal{HC}^1(\mathcal{A}_n, \mathcal{A}_n^*) = \{0\}$.

Proof Take a derivation $D: \mathcal{A}_n \to \mathcal{A}_n^*$. Then D is bounded because the semigroup algebra $\mathcal{A}_n$ is finite dimensional.

We have $D(\delta_e) = D(\delta_o) = 0$, and, by Lemma 1.2, we have

$$D(\delta_a) = \lambda_e\delta_e^* + \lambda_1\delta_a^* + \cdots + \lambda_{n-2}\delta_{a^{n-2}}^*$$

for some $\lambda_e, \lambda_1, \ldots, \lambda_{n-2} \in \mathbb{C}$.

We know that D is cyclic if and only if satisfies the equation:

$$\langle f, D(g)\rangle + \langle g, D(f)\rangle = 0 \quad (f, g \in \mathcal{A}) . \qquad (2.1)$$

Take $f = \delta_{a^k}$ and $g = \delta_a$ for $k = 0, \ldots, n-1$, where $a^0 = e$ and $\delta_{a^0} = \delta_e$. Then

$$\langle f, D(g)\rangle + \langle g, D(f)\rangle = \left\langle \delta_{a^k}, D(\delta_a)\right\rangle + \left\langle \delta_a, D(\delta_{a^k})\right\rangle$$
$$= \left\langle \delta_{a^k}, D(\delta_a)\right\rangle + \left\langle \delta_a, k\delta_{a^{k-1}}D(\delta_a)\right\rangle$$
$$= \left\langle \delta_{a^k}, D(\delta_a)\right\rangle + k\left\langle \delta_a, \delta_{a^{k-1}}D(\delta_a)\right\rangle$$

$$= (k+1)\langle \delta_{a^k}, D(\delta_a)\rangle ,$$

and so, by (2.1), we have

$$\langle \delta_{a^k}, \lambda_e \delta_e^* + \lambda_1 \delta_a^* + \cdots + \lambda_k \delta_{a^k}^* + \cdots + \lambda_{n-2} \delta_{a^{n-2}}^* \rangle = 0 ,$$

hence $\lambda_k = 0$ for all $k = 0, \dots, n-2$. So $D = 0$. Therefore $\mathcal{HC}^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$. Thus the theorem is proved.

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