Design of Linear Quadratic Regulator (LQR) Based on Solving Algebraic Riccati Equation (ARE) for Minimize the Performance Index.

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ملخص البحث

تتضمن هده الورقة التحليل النظري والحسابي لحل مشكلة المتحكم المنظم التربيعي الخطي LQR بهدف تصميم نظام حلقة مغلقة مستقرة دائما من خلال استخدام برنامج المحاكاة الماتلاب.

إن المتحكم المنظم التربيعي الخطي LQR هو نوع من انواع التحكم المثالي التي تتعامل مع انظمة التحكم المغلق لتقليل من مؤشر الاداء وبالتالي التقليل من الجهد والوقت والتكلفة وفق حدود وقيود محددة.

وبشكل اساسي ان هدا العمل يتضمن حل معادلة ريكيتة الجبرية ARE للحصول على المتحكم الامثل الدي يعطي مصفوفة محددة موجبة P التي تعطي ان النظام دائما مستقر وبالتالي وبشكل واضح يمكن الحصول على تصميم LQR قادر التقليل من مؤشر الاداء الي الحد الادنى.

هدا العمل قابل للتطبيق في تكنولوجيا إطلاق الصواريخ والمركبات الفضائية لتحقيق الحد الادنى من إستهلاك الوقود والتقليل من تكلفة التصميم.

تم شرح تركيب الهيكل التنظيمي للمتحكم المثالي LQR على نطاق واسع في هدا العمل .

في خلاصة هده الورقة قدمت طريقة تصميم تحكم مثالي LQR له القدرة على التقليل من مؤشر الاداء استنادا على نتائج حل معادلة ريكيتة الجبرية ARE، هده الطريقة تنتج نظام حلقة مستقر دائما.

ABSTRACT

This paper deals with the theoretical and computational analysis of linear quadratic regulator (LQR) problems, with the aim of providing solutions to them with MATLAB simulation and implementation. Linear quadratic regulator is a special type of optimal control problem that deals with linear system and minimization of cost or objective functions that are quadratic in

state and in control (subject to some constraints) with performance index J determination.
This work involves solving the associated algebraic Riccati equation (ARE) of the control systems and obtaining the optimal control gain. The work is mainly applicable in satellite and spacecraft launching technology to determine minimum time for spacecraft to meet its target, minimum fuel consumption for the operation of a rocket or spacecrafts and the minimum cost in the design of the spacecraft, through finding solution to the Algebraic Riccati Equation (ARE), whose solutions give the minimization parameters for the design and operation of the spacecraft. It is interesting to observe that minimizing the quadratic performance index J will always yields a stable closed loop system.
Structure of Q and R parameters are needed in the determination of optimal control gain of the systems, as they vary minimization of the quadratic performance index J. These structures are explicated extensively in this work.
If the main goal of the design procedure is solution to linear quadratic regulator problems is achieved through the use of the MATLAB, as it gives the positive definite matrix P, poles of the control system which shows that the system is always stable and the optimal control gain which shows that the cost or objective function is minimized.
In summary this paper has introduced a method of a control design that has the ability to minimize the performance index based J on (LQR).

Keywords: Numerical. Advanced Mathematical, Linear Quadratic Regulator (LQR), Algebraic Riccati Equation (ARE), Quadratic Optimal Control, MATLAB.

1. Introduction

Optimal control theory-which is playing an increasingly important role in the design of modern systems has as its objective the maximization of the return from, or the minimization of the cost of, the operation of physical, social, and economic processes.

Classical control system design is generally a trial-and-error process in which various methods of analysis are used iteratively to determine the design parameters of an "acceptable" system. Acceptable performance is generally defined in terms of time and frequency domain criteria such as rise time, Settling time, peak overshoot, gain and phase margin, and bandwidth.

Radically different performance criteria must be satisfied, however, by the complex, multiple-input, multiple-output systems required to meet the demands of modern technology.
For example, the design of a spacecraft attitude control system that minimizes fuel expenditure is not amenable to solution by classical methods, a new and direct approach to the synthesis of these complex systems, called optimal control theory, has been made feasible by the development of the digital computer.
The objective of optimal control theory is to determine the control signals that will cause a process to satisfy constraints and at the same time minimize (or maximize) some performance criterion.

2.Quadratic optimal regulator problems:

We shall now consider the optimal regulator problem that, given the system equation

$$\dot{x} = Ax + Bu \qquad (2\text{-}1)$$

Determines the matrix K of the optimal control vector

$$u(t) = -K\,x(t) \qquad (2\text{-}2)$$

So to minimize the performance index

$$J = \int_0^{\infty} (x^T Q x + u^T R u)\,dt. \qquad (2\text{-}3)$$

Where Q is positive-definite (or positive -semi definite) hermitian or real symmetric matrix and R is a positive-definite hermitian or real symmetric matrix. Note that the second term on the right-hand side of equation (2-3) accounts for the expenditure of the energy of the control signal. The matrices Q and R determine the relative importance of the error and the expenditure of this energy.
In this problem, we assume that the control vector u(t) is unconstrained.
As will be seen later, the linear control law given by equation (2-2) is the optimal control law. Therefore, if the unknown elements of the matrix K are determined so as to minimize the performance index, then u(t) = -kx(t) is optimal for any initial state x(0). The block diagram showing the optimal

configuration is shown in figure 2-1

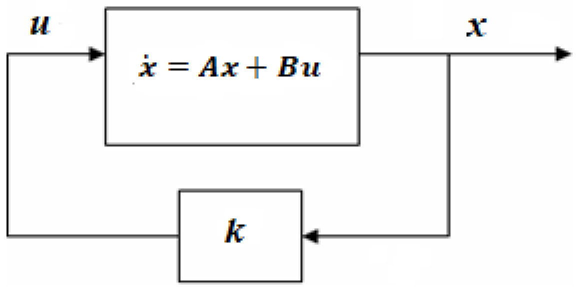

Figure 2-1: Optimal Regulator System.

Now let us solve the optimization problem. Substituting equation (2-2)
Into equation (2-1), we obtain

$$\dot{x} = Ax - BKx = (A - BK)x \qquad (2\text{-}4)$$

In the following derivations, we assume that the matrix A-BK is stable, or that the eigenvalues of A-BK have negative real parts.

Substituting Equation (2-2) into Equation (2-3) yields

$$J = \int_0^{\infty} (x^T Q x + x^T K^T R K x) dt$$

$$= \int_0^{\infty} x^T (Q + K^T R K) x dt \qquad (2\text{-}5)$$

Let us set

$$x^T (Q + K^T R K) x = -\frac{d}{dt}(x^T P x)$$

Where P is positive -definite hermitian or real symmetric matrix.
Then we obtain

$$x^T (Q + K^T R K) x = -x^T P x - x^T P x = -x^T [(A - BK)^T P + P(A - BK)] x$$

Comparing both sides of this last equation and noting that this equation must hold true for any x, we require that

$$(A - BK)^T PP(A - BK) = -(Q + K^T R K) \qquad (2\text{-}6)$$

it can be proved that if A-BK is a stable matrix, there exists a positive-definite matrix P that satisfies Equation (2-6).

Hence our procedure is to determine the element of P from Equation (2-6) and see if it is positive definite. (Note that more than one matrix P may satisfy this equation. If the system is stable, there always exists one positive-definite matrix P to satisfy this equation. This means that, if we solve this equation and find one positive-definite matrix P, the system is stable.

Other P matrices that satisfy this equation are not positive definite and must be discarded). The performance index J can be evaluated as

$$J = \int_0^{\infty} x^T(Q + K^T RK)x dt = -x^T Px = -x^T(\infty)P(\infty) + x^T(0)Px(0) \quad (2\text{-}7)$$

Since all eigenvalues of A-BK are assumed to have negative real, we havex (∞) → 0. Therefore, we obtain

$$J = x^T(0)Px(0) \quad (2\text{-}8)$$

Thus, the performance index J can be obtained in terms of the initial condition x(0) and P.

To obtain the solution to the quadratic optimal control problem, we proceed as follows: since R has been assumed to be a positive-definite Hermitian or real symmetric matrix, we can write

$$R = T^T T \quad (2\text{-}9)$$

Where T is a nonsingular matrix. Then Equation (2-6) can be written as

$$(A^T - K^T B^T)P + P(A - BK) + Q + K^T T^T TK = 0$$

Which can be rewritten as

$$A^T P + PA + [TK - (T^T)^{-1}B^T P]^T[TK - (T^T)^{-1}B^T P] - PBR^{-1}B^T P + Q = 0$$

The minimization of J with respect to K requires the minimization of

$$x^T[TK - (T^T)^{-1}B^T P]^T[TK - (T^T)^{-1}B^T P]x$$

with respect to K since this last expression is nonnegative, the minimum occurs when it is zero, or when

$$TK = (T^T)^{-1}B^T$$

Hence

$$K = T^{-1}(T^T)^{-1}B^T P = R^{-1}B^T P \quad (2\text{-}10)$$

Equation (2-10) gives the optimal matrix K. Thus the optimal control law to the quadratic optimal control problem when the performance index is given by Equation (2-3) is linear and is given by

$$u(t) = -Kx(t) = -R^{-1}B^T Px(t) \quad (2\text{-}11)$$

the matrix P in Equation (2-10) must satisfy Equation (2-6) or the following reduced equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \qquad (2\text{-}12)$$

Equation (2-12) is called the reduced-matrix Riccati equation. The design steps may be stated as follows:

Solve equation (2-12) is called the reduced-matrix Riccati equation, for the matrix P. (If a positive-definite matrix P, the system is stable, or matrix A-BK is stable).

Substitute this matrix P into Equation (2-10). The resulting matrix K is the optimal matrix. Note that if the matrix A-BK is stable, the present method always gives the correct result.

Finally, note that if the performance index is given in terms of the output vector rather than the state vector, that is

$$J = \int_0^\infty (Y^T QY + u^T Ru)$$

Then the index can be modified by using the output equation

$$Y = Cx \qquad (2\text{-}13)$$

To

$$J = \int_0^\infty (x^T Qx + u^T Ru)dt \qquad (2\text{-}14)$$

And the design steps presented in this section can be applied obtain the optimal matrix K.

Example 2.1

Consider the system described by

$$\dot{x} = Ax + Bu$$

Where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The performance index J is given by

$$J = \int_0^\infty (x^T Qx + u^T Ru)dt$$

Where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = [1]$$

Assume that the following control law is used.

u = - Kx

Determine the optimal feedback gain matrix K that minimizes the objective function J.

The optimal feedback gain matrix K can be obtained by solving the following Riccati equation for a positive-definite matrix P

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

The result is

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Substituting this P matrix into the following equation gives the optimal K matrix:

$$K = R^{-1} B^T P$$

$$K = [1][0 \quad 1]\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = [1 \quad 1]$$

Thus, the optimal control signal is given by

$$u = -Kx = -x_1 - x_2$$

3.MATLAB solution to quadratic optimal regulator problem

In MATLAB the command:

lqr(A,B,Q,R)

Can be used to solve the continuous-time linear quadratic regulator problem and the associated Riccati equation. The feedback control law assumed here is given below

u = - Kx

Such that the performance index is minimized.

$$J = \int_0^\infty (x^T Qx + u^T Ru)dt$$

Subject to the constraint equation

$$\dot{x} = Ax + Bu$$

Another command

$$[K,P,E] = lqr\ (A,B,Q,R) \qquad (3\text{-}1)$$

Returns the gain matrix K eigenvalue vector E and matrix P, the unique positive-definite solution to the associated matrix Riccati equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

If matrix A-BK is a stable matrix, such a positive-definite solution P always exists. The eigenvalue vector E given the closed loop poles of A-BK.

Let us consider example 2.1 again, the MATLAB code for this example is give below:

```
%--------Design of quadratic optimal regulator system ----
A = [0 1;0 -1];
B = [0;1];
Q = [1 0;0 1];
R = [1];
K = lqr(A,B,Q,R)
K = [1.0  1.0]
```

It is important to note that for certain system matrix A-BK cannot be made a stable matrix, whatever K is chosen. In such a case, the positive-definite matrix P that satisfies the Riccati equation will not exist either. In other words the commands:

$$K = lqr(A,B,Q,R)$$

$$[K,P,E] = lqr(A,B,Q,R)$$

Example 3.1

Consider the system defined by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u$$

Show that the system cannot be stabilized by the state-feedback control scheme

$$u = -kx$$

whatever matrix K is chosen. Define

$$K = [K_1 \qquad K_2]$$

Then

$$A - BK = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 - K_1 & 1 - K_2 \\ 0 & 2 \end{bmatrix}$$

Hence the characteristic equation becomes

$$|sI - A + BK| = \begin{bmatrix} s + 1 + K_1 & -1 + K_2 \\ 0 & s - 2 \end{bmatrix}$$

$$= (s + 1 + K1)(s - 2) = 0$$

The closed-loop poles are located at

$$s = -1\text{-}K_1,\ s = 2.$$

Since the pole at s = 2 is in the right-half s plant, the unstable whatever K matrix is chosen. Hence quadratic optimal control techniques cannot be applied to this system.

Example 3.2

Consider that the system defined by

$$\begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 0.1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Assume the following values for the performance index parameters Q and R

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ R = [1]$$

The MATLAB code for this example is given below: A=[0.1 -1;1 0]

B=[1;0]

Q =[1 0;0 1] R=[1]

[K,P,E]=lqr(A,B,Q,R)

Solution

$$K = [\,1.4559 \quad 0.4142\,]$$

$$P = \begin{bmatrix} 1.4559 & 0.4142 \\ 0.4142 & 1.0175 \end{bmatrix}$$

$$\lambda_1 = -0.6779 + j\,0.9770$$
$$\lambda_2 = -0.6779 - j\,0.9770$$

From which P the performance index J can be calculated as follows:

$$J = X^T(0)P(0)$$

When

$$X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$J = 1.4559$$

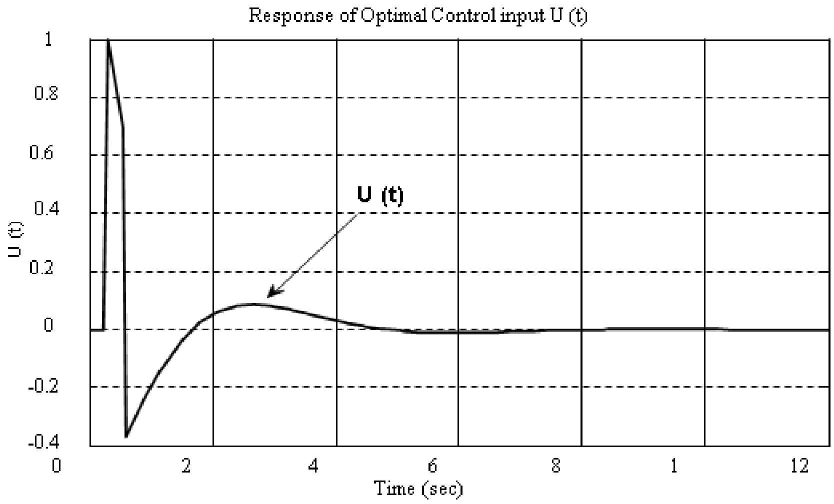

Figure 3-1: Closed loop System response (input signal is impulse).

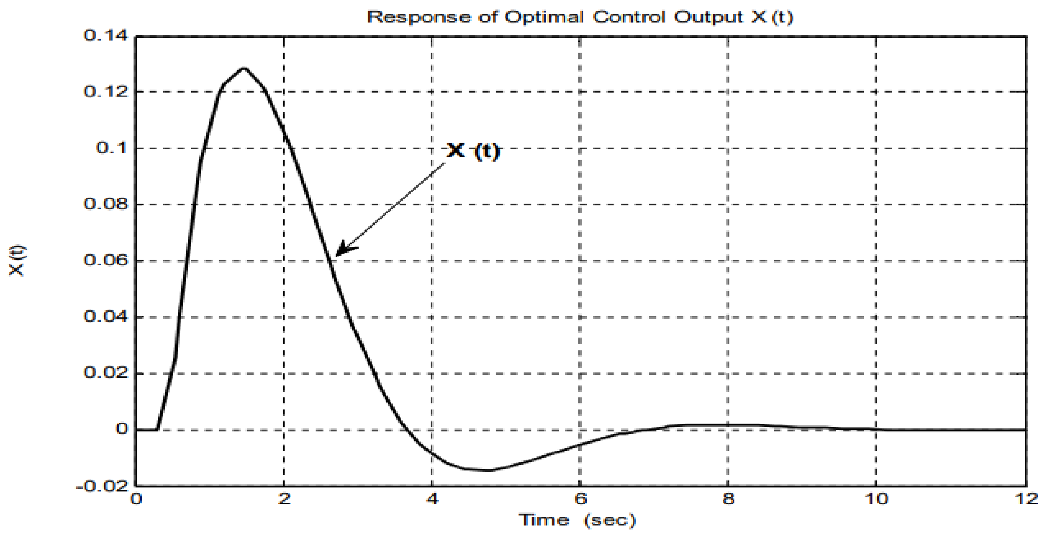

Figure 3-2: Closed loop System response (input signal is impulse).

Figure 3-1 and Figure 3-2 show the dynamic response of the system under optimal feedback law. The input signal considered her is a discrete impulse.

The results given in Figures 3-1 and Figure 3-2 were obtained with R = 1. Let us now consider other values for this tuning parameter:
R = 0.01, R = 0.1, R = 1, R = 10, and R = 100.

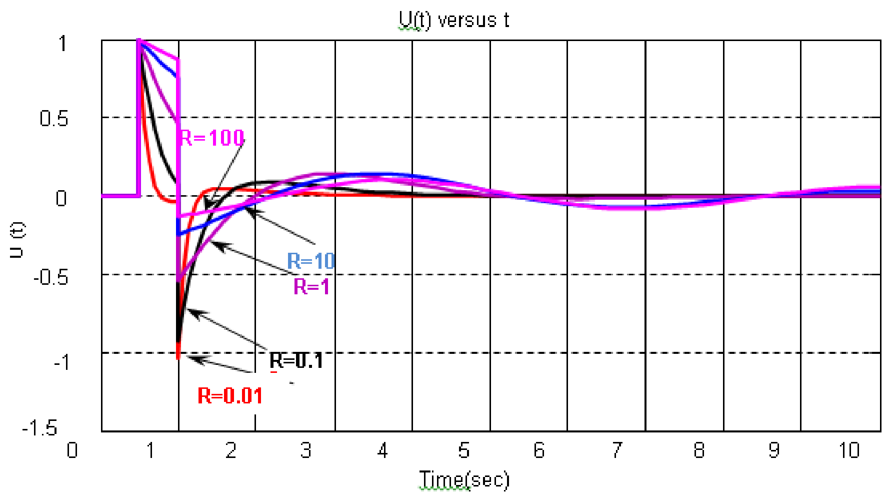

Figure 3-3: Closed loop System response with different values of R.

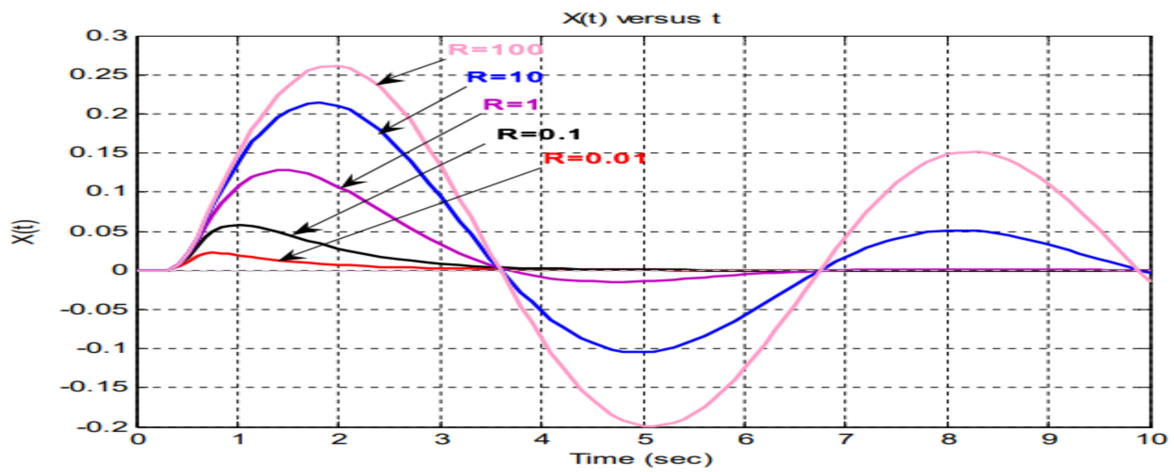

Figure 3-4: Closed loop System response with different values of R.

Figure 3-4 shows the state vector x(t) obtained with several values of R, all trajectories are optimal solution each for a different performance index. It is worthwhile nothing that as the value of R increased the area under the signal x(t) is increased.
In other words, the system performance becomes worst. This can explained by observing that increasing the value of R is equivalent to decreasing the value of

Q, which in turn means that the performance index J will be dominated by the term $u(t)^T Ru(t)$.
Thus, the obtained optimal feedback matrix K will serve the reduction of this last term rather than the other term $x(t)^T Qx(t)$.
Similarly, Figure 3-3 shows the optimal control signals u(t) obtained with several values of R, all trajectories are optimal solution each for a different performance index. It is worthwhile nothing that as the value of R increased the area under the signal u(t) is decreased.

In other words, the required control power is reduced. This is a very logic results since increasing the value of R implied that J is dominated by the term $u(t)^T Ru(t)$ as explained above.

4. Conclusion

This paper has considered the optimal control dynamic problem. It has shown that the optimal control K can be obtain by solving a Riccati equation, moreover, MATLAB command (LQR) can be used very efficiently to solve optimal controller problem.
It is interesting to observe that minimizing the quadratic performance index J will always yields a stable closed loop system.
The control law used here is u = -Kx, this requires that all states be available for feedback which is not always the case in real applications.
Nevertheless, the results given here is still of great practical interest for many other applications where this assumption is approved.
Moreover, if not all state variables are available for feedback, then, we need to employ a state observer to estimate all un measurable states, this topic is beyond the scope of this paper.
Finally, the results given here (i.e. example 3.2) have shown the clear contradiction between achieving best dynamic response and reducing the control energy. That is to say decreasing one of them will automatically increase the other (i.e. nothing is free). More important, the matrix R can be used as a tradeoff parameter between these conflicting objectives.
Finally this result shows the great power of (LQR) as a control design tool.

5. Recommendation

There are still fertile fields or areas yet to be explored in the research areas of optimal control. This has necessitated the interest of modern mathematicians and other pure science related field.

The concept of linear quadratic regulators or generally quadratic optimal control should be revisited from a multidisciplinary point of view to give a wider content of reference ideas and knowledge base; and also to enhance a richer and more accurate policy formulation. The results obtained can be made better through the use of modern simulation software. This will not enhance visualization of solution but also allow for deliberate error introduction for the extreme situation analysis testing.

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